MAXIMUM DRAG IN SUPERSONIC FLOW

(O MAKSIMAL'NOM SOPROTIVLENII V SVERKHZVUKOVOM POTOKE)

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The wave resistance of a body in steady supersonic gas flow is equal to zero, if this body does not produce shock waves, and the flow does not become discontinuous. An example of this is the Busemann biplane. A simple investigation, leaving out the detailed structure of the flow, enables us to find another upper bound for the wave resistance for given parameters of the body.

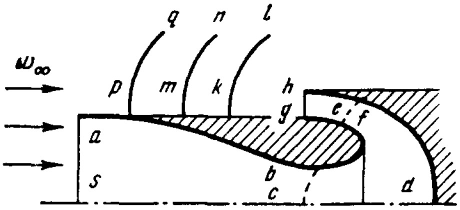

FIG. 1.

From the momentum theorem, there a relation between the interacting force of the gas jet on the body and the angle of deflection of the jet from the initial direction can be derived. For the maximum drag to occur, we must find the best method to turn the flow, drawing the greatest possible mass of gas into it. The solution of the problem may be based either on plane flow or on axisymmetric flow, as shown in Fig. 1. The gas trapped by the diffuser is ejected in a direction opposite to the incoming flow. The lines *kl* and *pg* are shock waves, and the line *mn* is the dividing streamline. This example shows that the body being sought need not be a continuous solid body, but the gas may flow through a control volume (e.g. *sagh*) enclosing it. In view of the complexity of the interaction of the two parts of the flow in the region *qpkl*, it is advisable as a first step to leave out the structure of the flow at the wall, and to obtain an estimate for the magnitude of the wave resistance.

The equations of gas dynamics are

$$\oint_L y^\nu \left[\rho uv\, dx - (p + \rho u^2)\, dy\right] = 0, \qquad \oint_L y^\nu \rho\, (v\, dx - u\, dy) = 0$$

$$\rho = \varphi W^{\frac{1}{\varkappa-1}}, \qquad p = \varphi W^{\frac{\varkappa}{\varkappa-1}} \qquad \left(\varphi = \frac{\rho^{\varkappa k}}{p^k}, \quad k = \frac{1}{\varkappa - 1}\right) \tag{1}$$

$$W = \frac{\varkappa+1}{2\varkappa} - \frac{\varkappa-1}{2\varkappa}(u^2 + v^2)$$

Here L is the boundary of an arbitrary flow region, which may be multiply-connected, x, y are the cartesian coordinates, u, v the components of the velocity vector in the x-y directions, respectively, relative to the critical velocity a_* , ρ is the density, relative to the density ρ_∞ upstream, p is the pressure, relative to $\rho_\infty a_*^2$; φ is the entropy function, κ is the adiabatic exponent and ν equals 0 or 1 respectively for plane or axisymmetric geometry.

Let the front part of the body, through the surface of which gas may flow, be bounded by the rectangle $0 \leqslant x \leqslant X$, $0 \leqslant y \leqslant Y$, where X, Y are given numbers. We select a control volume in the following manner. Let sa denote a Mach line of the uniform incoming flow starting from some point a. If the body is close to that shown in Fig. 1, then the point a will be the front point of a sharp profile. From this point attached shock waves may generate. If the body results in a detached shock wave, then we choose the point a to be the intersection of the shock wave with the streamline, which divides the gas flowing through the inside of the body. The remaining part of the contour through which gas may flow will be denoted by ah. The contour ash is closed by the axis of symmetry, and forms the body surface.

The arbitrariness in the choice of the control volume permits us, in particular, to explain the role of the bow shock in increased drag. If for the obtained maximum force on the body, it is necessary to influence the gas not passing through the shock, then the results of the solution of the variational problem permits further deductions on the drag estimate.

Parameters of the incoming flow will be given the subscript ∞. The equation of the line ah will be written as $y = f(x)$. Then, by virtue of the first equation in (1), the resultant pressure in the x-direction equals

$$\chi = \frac{(\varkappa + 1)\, y_a^{\nu+1}}{2\varkappa\,(\nu + 1)}\,(1 + w_\infty^2) + \int_{x_a}^{x_h} f^\nu\,[(p + \rho u^2)\,f' - \rho u v]\,dx \qquad (2)$$

Here w_∞ is the velocity of the incoming flow. To transform the first term of the right-hand side of (2), we use the last three equations of (1). If the total outflow of the gas across the contour sah equals zero, then the second equation of (1) gives

$$\Psi = 0 = \frac{w_\infty y_a^{\nu+1}}{\nu + 1} + \int_{x_a}^{x_h} f^\nu \rho\,(uf' - v)\,dx \qquad (3)$$

To pose the variational problem for the determination of the body with maximum drag, it is necessary, in addition to the functional (2) and condition (3), to consider the equations of gas dynamics, relations on any discontinuities that result, and the boundary conditions of the problem. Such a complete determination is not attempted in the present paper.

We shall consider the problem, based only on equalities (2) and (3), the conditions

$$0 \leqslant f(x) \leqslant Y \qquad \text{for } 0 \leqslant x \leqslant X \qquad (4)$$

and the obvious condition

$$\varphi \leqslant \varphi_\infty \qquad (5)$$

expressing the non-decrease of entropy across a shock wave. The solution of this problem may lead to such flow parameters on the line ah which cannot be realized. If in addition a front shock wave turns out to be unnecessary, then the value thus found must be considered as the upper limit of the drag.

The following variational problem arises. It is required to find the functions $f(x)$, $u(x)$, $v(x)$ and $\varphi(x)$, for which the maximum of the functional (2) is obtained, under conditions (3) – (5) and given values of w_∞, X, Y.

We construct the functional

$$J = \chi + \lambda\Psi$$

$$J = \frac{F}{\nu + 1} y_a^{\nu+1} + \int_{x_a}^{x_h} \Phi(f, f', w, \vartheta, \varphi)\, dx, \qquad F = \frac{(\varkappa + 1)(1 + w_\infty^2)}{2\varkappa} + \lambda w_\infty$$

$$\Phi = f^\nu [(p + \rho w^2 \cos^2\vartheta) f' - \rho w^2 \sin\vartheta \cos\vartheta - \lambda\rho w (\sin\vartheta - f' \cos\vartheta)]$$

Here λ is a constant Lagrange multiplier, w is the modulus of the velocity, and θ is the angle of inclination of the velocity to the x-axis.

We first assume that $\varphi = \varphi_\infty$, and calculate the first variation

$$\delta J = (Fy^\nu - \Phi_{f'})_a \delta y_a - \Phi_a \delta x_a + (\Phi_{f'})_h \delta y_h + \int_{x_a}^{x_h} \left[\left(\Phi_f - \frac{d}{dx}\Phi_{f'}\right)\delta f + \Phi_w \delta w + \Phi_\vartheta \delta\vartheta\right] dx \qquad (6)$$

Here the subscripts f, f', w, ϑ denote partial derivatives

$$(Fy^\nu - \Phi_{f'})_a = y_a^\nu [F - (p + \rho w^2 \cos^2\vartheta + \lambda\rho w \cos\vartheta)_{ah}]$$

$$(\Phi_{f'})_h = y_h^\nu (p + \rho w^2 \cos^2\vartheta + \lambda\rho w \cos\vartheta)_h$$

The double subscript ah indicates that a quantity is taken at the point a as it is approached from the point h.

Other quantities appearing in equation (6) have the form

$$\Phi_f - \frac{d}{dx}\Phi_{f'} = \frac{\nu\Phi}{f} - \frac{d}{dx}[f^\nu(p + \rho w^2 \cos^2\vartheta + \lambda\rho w \cos\vartheta)]$$

$$\Phi_w = -\frac{f^\nu \rho}{\varkappa + 1 - (\varkappa - 1) w^2} \{2w(\varkappa + 1 - \varkappa w^2)(\sin\vartheta - f'\cos\vartheta)\cos\vartheta + [\varkappa + 1 - (\varkappa - 1) w^2] w f' + \lambda(\varkappa + 1)(1 - w^2)(\sin\vartheta - f'\cos\vartheta)\}$$

$$\Phi_\vartheta = -f^\nu \rho\, [w^2(\cos 2\vartheta + f' \sin 2\vartheta) + \lambda w(\cos\vartheta + f'\sin\vartheta)]$$

The necessary condition for the maximum χ is $\delta J \leqslant 0$ for the admissible variations. From (6), it is clear that this condition is satisfied when

$$x_a = 0, \qquad y_h = Y \qquad (7)$$

and if the following conditions hold

$$\Phi_a \geqslant 0, \qquad (\Phi_{f'})_h \geqslant 0 \qquad (8)$$

$$(Fy^\nu - \Phi_{f'})_a = 0 \qquad (9)$$

$$\Phi_f - \frac{d}{dx}\Phi_{f'} = 0, \quad \Phi_w = 0, \quad \Phi_\vartheta = 0 \qquad (0 \leqslant x \leqslant X) \qquad (10)$$

Inequalities (8) ensure $\delta J \leqslant 0$ on the grounds that the admissible variations satisfying equation (7) also satisfy the conditions $\delta x_a \geqslant 0$, $\delta y_h \leqslant 0$.

The functions $f(x)$, $w(x)$ and $\vartheta(x)$ with $0 \leqslant x \leqslant X$ and the quantity y_a are found from equations (10) and (9). λ is found from equation (3). After this, condition (8) must be verified.

One particular solution in the plane and axisymmetric cases is given by

$$(f')^{-1} = 0, \qquad w = 1, \qquad \vartheta = \pi$$

In this case, the gas is turned round against the incoming flow at sonic velocity. In actuality, such a flow can be realized for a finite-dimensional body.

In the plane case, $\nu = 0$, the quantity Φ_f is zero, and the first equation in (10) gives

$$p + \rho w^2 \cos^2 \vartheta + \lambda \rho w \cos \vartheta = \text{const}$$

This equation, together with the second and third equations of (10), shows that in plane flow the quantities f', w and ϑ for $0 \leqslant x \leqslant X$ are all constant.

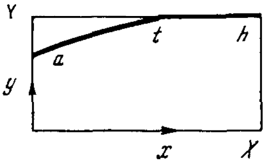

FIG. 2.

Another particular solution for plane flow is given by

$$f(x) = Y, \qquad \vartheta = \arc\cos\left[-\left(\frac{(\varkappa + 1)(w^2 - 1)}{2w^2}\right)^{1/2}\right] \tag{11}$$

To find the necessary quantities, the values of w and Y are given under the condition $1 \leqslant w \leqslant \sqrt{(\varkappa + 1)/(\varkappa - 1)}$, the quantities f and ϑ are found from (11), w_∞ from (9), and $X = x_h$ is calculated from equation (3) for $y_a = Y$ and $x_a = 0$. This solution stipulates the maximum mass of the gas trapped by the diffusor, which then turns it into the opposite direction.

In the axisymmetric case, the solution determined by equation (10) does not exist for all values of w_∞, X and Y. The line ah being sought may consist of two pieces (Fig. 2). On the part at a two-sided extremum is realized, while the piece th is defined by the equation $f(x) = Y$ for a single-sided admissible variation $\delta f \leqslant 0$. In this case, the functional J is written as

$$J = \frac{1}{2} F y_a^2 + \int_{x_a}^{x_t} \Phi dx + \int_{x_t}^{x_h} \Phi \, dx$$

The first variation of this functional equals

$$\delta J = (F y^2 - \Phi_{f'})_a \, \delta y_a - \Phi_a \delta x_a + (\Phi_{ta} - \Phi_{th}) \, \delta x_t + (\Phi_{f'})_h \, \delta y_h + \int_{x_a}^{x_t} \left[\left(\Phi_t - \frac{d}{dx} \Phi_{f'}\right) \delta f + \Phi_w \delta w + \Phi_\vartheta \delta \vartheta\right] dx + \int_{x_t}^{x_h} \left[\left(\Phi_f - \frac{d}{dx} \Phi_{f'}\right) \delta f + \Phi_w \delta w + \Phi_\vartheta \delta \vartheta\right] dx$$

Here the double subscripts indicate that the quantities are taken at the point t as it is approached from the side of the points specified by the second subscript. Functions $w(x)$ and $\vartheta(x)$ may have discontinuities at the point t.

To ensure the inequality of $\delta J \leqslant 0$ we require the following conditions to be satisfied

$$\Phi_t \leqslant 0, \qquad (\Phi_{f'})_h \geqslant 0 \tag{12}$$

$$(Fy^2 - \Phi_{f'})_a = 0 \tag{13}$$

$$\Phi_{ta} - \Phi_{th} = 0 \tag{14}$$

$$\Phi_f - \frac{d}{dx}\Phi_{f'} = 0, \quad \Phi_w = 0, \quad \Phi_\vartheta = 0 \qquad (0 \leqslant x \leqslant x_t) \tag{15}$$

$$f(x) = Y, \quad \Phi_w = 0, \quad \Phi_\vartheta = 0 \qquad (x_t \leqslant x \leqslant X) \tag{16}$$

$$\Phi_f - \frac{d}{dx}\Phi_{f'} \geqslant 0 \qquad (x_t \leqslant x \leqslant X) \tag{17}$$

The second and third equations in (16), apart from w and ϑ, contain only constant quantities. From this it follows that w and ϑ are constant on the segment th.

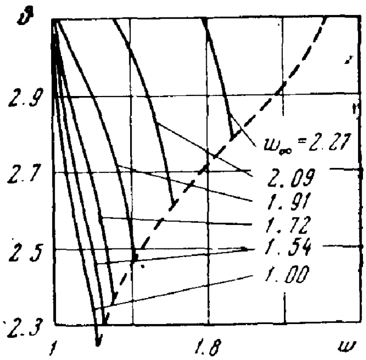

FIG. 3.

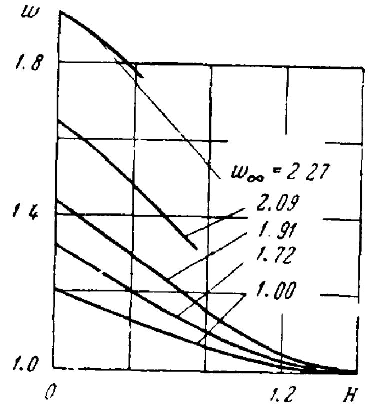

FIG. 4.

For given values of w_∞, X and Y, the quantities λ, y_a, x_t and the functions $f(x)$, $w(x)$, and $\vartheta(x)$ in the interval at, and the quantities f, w and v in th are determined by equations (3), (13) – (16), and the boundary condition $f(x_i) = Y$. After determining them, it is necessary to verify that conditions (12) and (17) are satisfied. Condition (17) can be simplified with the help of the relations $f(x) = Y$, $w(x) = \text{const}$, and $\vartheta(x) = \text{const}$ and assumes the form

$$\rho w \sin\vartheta\,(w\cos\vartheta + \lambda) \leqslant 0 \tag{18}$$

A particular solution of this type is obtained for $x_t = 0$ when all the gas entering the inside of the body is ejected across a cylindrical lateral surface. In this case, conditions (3), (12), (13), (16) and (17) must be satisfied.

Let there now be an admissible entropy increase $\varphi \leqslant \varphi_\infty$. We introduce the notation $\varphi = \varphi_\infty \varphi_*$, where $0 < \varphi_* \leqslant 1$. In expression (6), there now appears for the first variation δJ an additional term

$$\delta J_* = \int_{x_a}^{x_h} \Phi \frac{\delta\varphi_*}{\varphi_*}\,dx$$

We substitute into this expression the functions found from the solution with $\varphi = \varphi_\infty$. The assumed variation $\delta\varphi_*$ satisfies the condition $\delta\varphi_* \leqslant 0$. Consequently, the condition $\delta J_* \leqslant 0$ is satisfied for

$$\Phi \geqslant 0 \qquad (0 \leqslant x \leqslant X) \tag{19}$$

Thus, after finding the solution from equations (3), (9) and (10) or from (3), (13) – (16), it is necessary to verify that conditions (8), (19), or (12), (18) and (19) are satisfied. The

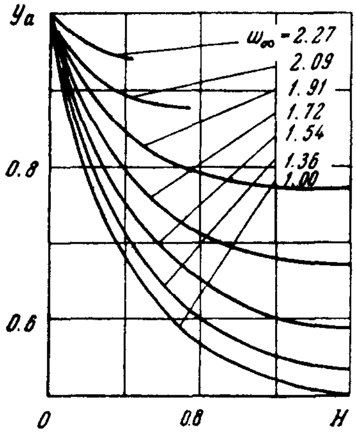

FIG. 5.

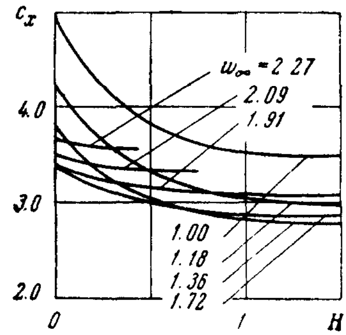

FIG. 6.

satisfaction of the first condition of (8) is then ensured by the satisfaction of condition (19).

Numerical examples for equations (9) and (10) were carried out with $\kappa = 1.4$ for plane and axisymmetric flows. For all values of w_∞ in the supersonic interval $1 \leqslant w_\infty \leqslant \sqrt{(\kappa + 1)/(\kappa - 1)}$ and for all values of the ratio $L = X/Y$ in the interval $0 \leqslant L < \infty$ conditions (8) and (19) are satisfied. From this, it follows that at least for $\kappa = 1.4$, the maximum drag is obtained by the flow of a gas past a body without any shock waves.

The computed results for the plane problem are given in Figs. 3–6. In all cases, the parameter for the curves is the incoming velocity w_∞. In Fig. 3, the full lines indicate the dependence of ϑ on w, and the dashed line gives the locus of the points of the particular solution (11). In Figs. 4–6, the dependence of the quantities w, y_a, and the coefficient of wave resistance c_x on the quantity H are given, where

$$c_x = 2\chi / w_\infty^2 Y, \qquad H = \text{arc tan } f'$$

In axisymmetric flow, the coefficient c_x is defined by the formula

$$c_x = 4\chi / w_\infty^2 Y^2$$

Its value agrees with c_x in plane flow, when $H = 0$ or $H = \pi/2$, where

$$H = \text{arc tan} \frac{y_h - y_a}{x_h - x_a}$$

In the intermediate cases, the coefficient of wave resistance of a body of revolution, within the accuracy of Fig. 6, does not differ from the values of c_x for plane bodies.

We remark that for a Mach number of the incoming flow $M = 4$, for example, the maximum drag of a body of revolution may be twice as large as the wave resistance of a semi-infinite cylinder with plane nose section in the case of axisymmetry. To make such comparisons, we used the calculations of axisymmetric flow with detached shocks given in [1].

BIBLIOGRAPHY

1. Belotserkovskii, O.M. Simmetrichnoe obtekanie zatuplennykh tel sverkhzvukovym porokom sovershennovo i real'novo gaza. (Symmetric flow of ideal and real gases past a blunt body). *Zhur. vychisl. matem. i matem. fiz.*, 1962, Vol. 2, No. 6.

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